

Differential Transformation Method For Solving Interval Differential Equations

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ABSTRACT: In this paper, we present Differential Transformation Method (DTM) for solving Interval Differential Equations (IDEs). The proposed method is also illustrated by some examples. And some error comparison is made with Runge-Kutta method of order 4 (RK4).

Keywords: Interval-valued functions, Interval differential equations, Hukuhara differentiability, Differential transformation method.

INTRODUCTION

The interval-valued analysis and interval differential equations (IDEs) are the particular cases of the set-valued analysis and set differential equations, respectively. Some systematic studies in these area are contained in (Aubin and Frankowska, 1990; Lakshmikantham et al., 2006; Moore, 1966). Stefanini and Bede (Stefanini and Bede, 2009) started the research of IDEs with two different concepts of the Hukuhara derivative. The first concept is a classical one, i.e, the derivative is taken from the paper of Hukuhara (Hukuhara, 1967), while the second one is a new form of differentiation of interval-valued mappings (Stefanini and Bede, 2009). The methods used in the study of interval differential equations are similar to those used to study fuzzy differential equations. However, some of the results are particular for IDEs, or have much simpler formulation in interval setting than corresponding theorems in fuzzy setting. The reason for this is that the space of intervals is separable, so it is locally compact, while the set of fuzzy numbers is non separable, so it is not locally compact. So, when we study fuzzy analysis we cannot be sure that a closed ball is compact, while, in the interval case this is obvious. In this paper, we apply DTM for solving IDEs, based on Hukuhara derivative.

Preliminaries

Definitions and notations

Let E denote the family of all nonempty, compact and convex subsets of the real line \mathbb{R} (intervals). The addition and scalar multiplication in E , we define as usual, i.e. for

$A, B \in E, A = [a^-, a^+], B = [b^-, b^+], a^- \leq a^+, b^- \leq b^+$, and $\lambda \geq 0$ we have

$A + B = [a^- + a^+, b^- + b^+], \lambda A = [\lambda a^-, \lambda a^+], (-\lambda)A = [-\lambda a^+, -\lambda a^-]$.

Note that for $A \in E, \lambda_1, \lambda_2, \lambda_3, \lambda_4 \in \mathbb{R}, \lambda_3 \lambda_4 \geq 0$ it holds:

$\lambda_1(\lambda_2 A) = (\lambda_1 \lambda_2)A$ and $(\lambda_3 + \lambda_4)A = \lambda_3 A + \lambda_4 A$.

The Hausdorff metric H in E is defined as follows:

$H(A, B) = \max\{|a^- - b^-|, |a^+ - b^+|\}$

for $A = [a^-, a^+], B = [b^-, b^+]$. It is known (Stefanini and Bede, 2009) that (E, H)

is a complete, separable and locally compact metric space. For The metric H the following properties hold (Lakshmikantham et al., 2006)

$H(A + C, B + C) = H(A, B)$

$H(A + B, C + D) \leq H(A, C) + H(B, D)$,

$H(\lambda A, \lambda B) = |\lambda| H(A, B)$,

for $A, B, C, D \in E, \lambda \in \mathbb{R}$. Let $A, B \in E$. If there exists an interval $C \in E$ such that $A = B + C$, then we call C the Hukuhara difference of A and B . The interval C denoted by $\ominus B$. Note that $A \ominus B \neq A + (-1)B$.

For $A = [a^-, a^+] \in E$ denote the length and the magnitude of A by $\text{len}(A) := a^+ - a^-$ and $\|A\| := H(A, \{0\}) = \max\{|a^+|, |a^-|\}$, respectively. It is known that $A \ominus B$ exists in the case $\text{len}(A) \geq \text{len}(B)$ (Lakshmikantham *et al.*, 2006; Stefanini and Bede, 2009). Also one can verify the following properties for $A, B, C, D \in E$:

If $A \ominus B, A \ominus C$ exists, then $H(A \ominus B, A \ominus C) = H(B, C)$;

If $A \ominus B, C \ominus D$ exists, then $H(A \ominus B, C \ominus D) = H(A + D, B + C)$;

If $A \ominus B, A \ominus (B + C)$ exists, then there exists $(A \ominus B) \ominus (A \ominus C)$ and $(A \ominus B) \ominus (A \ominus C) = C \ominus B$.

Definition 2.1. We say that the interval-valued mapping $F: [\alpha, \beta] \rightarrow E$ if for every $\epsilon > 0$ there exists $\delta = \delta(t, \epsilon) > 0$ such that, for all $s \in [\alpha, \beta]$ such that $|t - s| < \delta$, one has $H(F(t), F(s)) < \epsilon$. If $F: [\alpha, \beta] \rightarrow E$ is continuous at every point $t \in [\alpha, \beta]$, then we will say that F is continuous on $[\alpha, \beta]$.

Definition 2.2. A mapping $F: [\alpha, \beta] \rightarrow E$ is Hukuhara differentiable at $t_0 \in [\alpha, \beta]$ if there exists $f'(t_0) \in E$ such that the limits

$$\lim_{h \rightarrow 0^+} \left(\frac{1}{h}\right) (f(t_0 + h) \ominus f(t_0)), \quad \lim_{h \rightarrow 0^+} \left(\frac{1}{h}\right) (f(t_0) \ominus f(t_0 - h)),$$

exist and are equal to $f'(t_0)$. The interval $f'(t_0)$ is said to be Hukuhara derivative of interval-valued mapping f at the point t_0 .

2.2 Interval differential equations

In this section we consider an interval-valued differential equation

$$y' = f(x, y), \quad y(x_0) = y_0$$

where $f: [a, b] \times E \rightarrow E$ with $f(x, y) = [f^-(x, y), f^+(x, y)]$ for $y \in E$

$$y = [y^-, y^+], \quad y_0 = [y_0^-, y_0^+].$$

We consider only H -differentiable solutions, i.e. there exists $\delta > 0$ such that there are no switching points in $[x_0, x_0 + \delta]$.

Differential transformation method

The basic definitions and fundamental operations of the differential transform are introduced in (Chen and Ho, 1999). The differential transform of the function $u(x)$ is the following form

$$U(k) = \frac{1}{k!} \left[\frac{d^k u(x)}{dx^k} \right]_{(x_0)}, \tag{2.1}$$

where $u(x)$ is the original function and $U(k)$ is the transformed function.

The inverse differential transform of $U(k)$ is defined as

$$u(x) = \sum_{k=0}^{\infty} U(k) (x - x_0)^k. \tag{2.2}$$

When (x_0) are taken as (0) , the function $u(x)$, (2.2), is expressed as the following

$$u(x) = \sum_{k=0}^{\infty} \frac{1}{k!} \left[\frac{d^k u(x)}{dx^k} \right]_{(0)} x^k. \tag{2.3}$$

Eq. (2.3) implies that the concept of differential transform is derived from Taylor series expansion.

In this paper, the lower case letters represent the original function and upper case letters stands for the transformed function (T-function). The fundamental mathematical operations performed by differential transform method can readily be obtained and are listed in Table 1.

By continuity condition, we define the differential transform of interval function

$x(t, r) = (\underline{x}(t, r), \bar{x}(t, r))$ as follows:

$$\underline{X}(k, r) = \frac{1}{k!} \left[\frac{d^k \underline{x}(t, r)}{dt^k} \right]_{(t_0)}, \tag{2.4}$$

Table 1: The original function and transformed function

Original function	Transformed function
$u(x) = f(x) \pm g(x)$	$U(k) = F(k) \pm G(k)$
$u(x) = \lambda f(x), \lambda \in \mathbb{R}$	$U(k) = \lambda F(k)$
$u(x) = x^m$	$\delta(k - m) = \begin{cases} 1, & k = m \\ 0, & \text{otherwise} \end{cases}$
$u(x) = \frac{d^r f(x)}{dx^r}, r \in \mathbb{N}$	$U(k) = (k + 1) \cdots (k + r) F(k + r)$
$u(x) = f(x) g(x)$	$U(k) = \sum_{r=0}^k F(r) G(k + r)$

$$\bar{X}(k, r) = \frac{1}{k!} \left[\frac{d^k \bar{x}(t, r)}{dt^k} \right]_{(t_0)}. \tag{2.5}$$

The inverse differential transform of $\underline{X}(k, r)$ and $\bar{X}(k, r)$ are defined respectively as

$$\underline{x}(t, r) = \sum_{k=0}^{\infty} \underline{X}(k, r)(t - t_0)^k, \tag{2.6}$$

and

$$\bar{x}(t, r) = \sum_{k=0}^{\infty} \bar{X}(k, r)(t - t_0)^k, \tag{2.7}$$

When t_0 are taken as (0), the functions $\underline{x}(t, r)$ and $\bar{x}(t, r)$ of (2.4) and (2.5), are expressed as the following

$$\underline{X}(k, r) = \frac{1}{k!} \left[\frac{d^k \underline{x}(t, r)}{dt^k} \right]_{(0)}, \tag{2.8}$$

$$\bar{X}(k, r) = \frac{1}{k!} \left[\frac{d^k \bar{x}(t, r)}{dt^k} \right]_{(0)}. \tag{2.9}$$

Eqs. (2.8) and (2.9) imply that the concept of differential transform is derived from Taylor series expansion. In this paper, the lower case letters represent the original function and upper case letters stands for the transformed function (T-function).

EXAMPLES

Example1. Consider interval differential equation

$$\begin{cases} y'(x) = -y(x) + [1,2]x, & x \in [0,1.5], \\ y(0) = [0,1]. \end{cases}$$

We denote $y = [\underline{y}, \bar{y}]$.

Or

$$\begin{aligned} \underline{y}'(x) &= -\bar{y} + x, \\ \bar{y}'(x) &= -\underline{y} + 2x, \\ \underline{y}(0) &= 0, \quad \bar{y} = 1. \end{aligned} \tag{3.1}$$

Taking differential transform of (3.1), we find

$$\begin{aligned} (k + 1)\underline{Y}(k + 1) &= -\bar{Y}(K) + \delta(K - 1), \\ (k + 1)\bar{Y}(k + 1) &= -\underline{Y}(k) + 2\delta(k - 1). \end{aligned} \tag{3.2}$$

From the initial conditions, we can write

$$\underline{Y}(0) = 0, \quad \bar{Y} = 1. \tag{3.3}$$

Substituting Eqs. (3.3) in (3.2), all spectra can be found as

$$\begin{aligned} \underline{y}(x) &= -x + \frac{1}{2}x^2 - \frac{1}{2}x^3 + \frac{1}{4!}x^4 - \frac{1}{40}x^5 + \dots \\ \bar{y}(x) &= 1 + \frac{3}{2}x^2 - \frac{1}{3!}x^3 + \frac{1}{8}x^4 - \frac{1}{5!}x^5 + \dots \end{aligned}$$

and we find the exact solution as follows:

$$y(x) = (2x - e^x + 2e^{-x} - 1, x + e^x + 2e^{-x} - 2), \tag{3.4}$$

The exact and approximate solutions are compared and plotted for $x \in [0,1.5]$ in Figure1.

Bound of Error = max{Error of $\underline{y}(t, r)$, Error of $\bar{y}(t, r)$ } is used in these examples.

Tabel 2 shows the errors in the solution by the differential transformation method of various orders, along with the result obtained by the Runge-Kutta method of order 4.

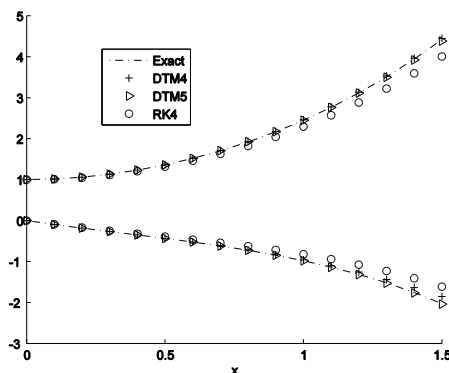


Figure 1. The exact and approximate solutions for $x \in [0,1.5]$

Example 2. Consider interval differential equation

$$\begin{cases} y'(x) = -y(x) + [1,2] \sin x, & x \in [0, \frac{\pi}{2}], \\ y(0) = [1,3], \end{cases}$$

Or

$$\begin{aligned} \underline{y}'(x) &= -\underline{y} + \sin x, \\ \overline{y}'(x) &= -\overline{y} + 2\sin x, \\ \underline{y}(0) &= 1, \quad \overline{y} = 3. \end{aligned} \tag{3.5}$$

Taking differential transform of (3.5), we have

$$\begin{aligned} (k+1)\underline{Y}(k+1) &= -\underline{Y}(k) + \frac{1}{k!} \sin\left(\frac{\pi k}{2}\right), \\ (k+1)\overline{Y}(k+1) &= -\overline{Y}(k) + \frac{2}{k!} \sin\left(\frac{\pi k}{2}\right). \end{aligned} \tag{3.6}$$

From the initial conditions, we can write

$$\underline{Y}(0) = 1, \quad \overline{Y} = 3. \tag{3.7}$$

Substituting Eqs. (3.7) in (3.6), all spectra can be found as

Table 2. Bounds of computational errors y of Example 1

x	DTM4	DTM5	DTM 6	RK4
0.0	0.00000e+0	0.00000e+0	0.00000e+0	0.00000e+0
0.1	2.48670e-7	4.14757e-9	5.92843e-11	5.17108e-3
0.2	7.91867e-6	2.64316e-7	7.55976e-9	1.18868e-2
0.3	5.98662e-5	2.99894e-6	1.28713e-7	2.04049e-2
0.4	2.51272e-4	1.67897e-5	9.61126e-7	3.10191e-2
0.5	7.64118e-4	6.38401e-5	4.56933e-6	4.40643e-2
0.6	1.89553e-3	1.90073e-4	1.63282e-5	5.99212e-2
0.7	4.08627e-3	4.78065e-4	4.79179e-5	7.90224e-2
0.8	7.94967e-3	1.06286e-3	1.21756e-4	1.01859e-1
0.9	1.43013e-2	2.15068e-3	2.77154e-4	1.28989e-1
1.0	2.41896e-2	4.04071e-3	5.78502e-4	1.61043e-1
1.1	3.89280e-2	7.14994e-3	1.12577e-3	1.98737e-1
1.2	6.01285e-2	1.20413e-2	2.06770e-3	2.42878e-1
1.3	8.97372e-2	1.94555e-2	3.61790e-3	2.94383e-1
1.4	1.30073e-1	3.03459e-2	6.07439e-3	3.54283e-1
1.5	1.83866e-1	4.59181e-2	9.84281e-3	4.23746e-1

$$\underline{y}(x) = 1 - 3x + x^2 - \frac{5}{3!}x^3 + \frac{1}{4!}x^4 - \frac{1}{40}x^5 + \dots$$

$$\overline{y}(x) = 3 - x + \frac{5}{2}x^2 - \frac{1}{3}x^3 + \frac{1}{8}x^4 - \frac{1}{5!}x^5 + \dots$$

and we find the exact solution as follows:

$$y(x) = \left(\frac{3}{2} \cosh x - 4 \sinh x - \frac{1}{2} \cos x + \sin x, 4 \cosh x - \frac{3}{2} \sinh x - \cos x + \frac{1}{2} \sin x\right),$$

The exact and approximate solutions are compared and plotted at $x \in [0, \frac{\pi}{2}]$ in Figure 2.

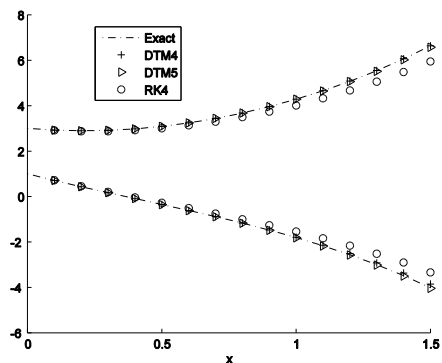


Figure 2. The exact and approximate solutions for $x \in [0, \frac{\pi}{2}]$

Table 3 shows the errors in the solution by the differential transformation method of various orders, along with the result obtained by the Runge-Kutta method of order 4.

Table 3. Bounds of computational errors y of Example 2

x	DTM4	DTM5	DTM 6	RK4
0.0	0.00000e+0	0.00000e+0	0.00000e+0	0.00000e+0
0.1	2.47321e-7	6.90550e-9	9.89666e-11	1.01753e-2
0.2	7.83486e-6	4.39554e-7	1.26391e-8	2.28956e-2
0.3	5.89405e-5	4.98055e-6	2.15497e-7	3.85623e-2
0.4	2.46233e-4	2.78425e-5	1.61126e-6	5.76265e-2
0.5	7.45517e-4	1.05693e-4	7.66928e-6	8.05956e-2
0.6	1.84184e-3	3.14122e-4	2.74353e-5	1.08040e-1
0.7	3.95554e-3	7.88543e-4	8.05914e-5	1.40599e-1
0.8	7.66877e-3	1.74948e-3	2.04951e-4	1.78995e-1
0.9	1.37529e-2	3.53219e-3	4.66874e-4	2.24036e-1
1.0	2.31973e-2	6.62060e-3	9.75102e-4	2.76632e-1
1.1	3.72403e-2	1.16857e-2	1.89850e-3	3.37804e-1
1.2	5.74019e-2	1.96280e-2	3.48826e-3	4.08700e-1
1.3	8.55205e-2	3.16254e-2	6.10502e-3	4.90612e-1
1.4	1.23792e-1	4.91847e-2	1.02515e-2	5.84986e-1
1.5	1.74815e-1	7.41983e-2	1.66114e-2	6.93452e-1

CONCLUSION

In this work, we introduced differential transformation method for approximate solution of a interval differential equations and illustrated by some numerical examples. The results showed that the DTM is remarkably effective and very simple.

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