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Product of Hall π -subgroups

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ABSTRACT: Recall that a finite group is a D_π - groups if evety π - subgroup is contained in a Hall π - subgroup and any two Hall π - subgroups are conjugate.In this paper we show that if finite group G=AB be the product of two subgroups A and B. If A,B, and G are D_π - group, for a set π of primes, then there exist Hall π -subgroups A_0 of A and B_0 of B such that A_0B_0 is a Hall π -subgroups of G.

Keywords: Hall π - subgroups, Finite group, D_π - group, product group

INTRODUCTION

In 1940 G. Zappa(see [24]) and in 1950 J.Szip (see [23]) studied bout products of groups concerned finite groups. In 1961 O.H.Kegel (see [8]) and in 1958 H.Wielandt (see [10]) expressed the famous theorem, whose states the solubility of all finite products of two nilpotent groups .

In 1955 N.Itô (see [7]) found an impressive and very satisfying theorem for arbitrary factorized groups. He proved that every product of two abelian groups is metabelian. Besides that, there were only a few isolated papers dealing with infinite factorized groups. (P.M. Cohn ,1956) (see[21]) and L.Redei ,1950)(see [22]) considered products of cyclic groups, and around 1965 O.H.Kegel (See [30] & [31]) looked at linear and locally finite factorized groups.

In 1968 N.F. Sesekin (see [19]) proved that a product of two abelian subgroups with minimal condition satisfies also the minimal condition . He and Amberg independently obtained a similar result for the maximal condition around 1972 (See [20]&[1]). Moreover, a little later the proved that a soluble product of two nilpotent subgroups with maximal condition likewise satisfies the maximal condition, and its Fitting subgroups inherits the factorization. Subsequently in his (Habilitationsschrift ,1973) he started a more systematic investigation of the following general question. Given a (soluble) product G of two subgroups A and B satisfying a certain finiteness condition x, when does G have the same finiteness condition x ?(see [20])

For almost all finiteness conditions this question has meanwhile been solved. Roughly speaking, the answer is 'yes' for soluble (and even for soluble-by-finite) groups. This combines theorems of B. Amberg (see [1], [2],[3],[4] and [6]) , N.S. Chernikov (see [5]), S. Franciosi, F. de Giovanni (see [3],[6],[32],[33],[34],[35], and [36]), O.H.Kegel (see [8]), J.C.Lennox (see [12]) , D.J.S. Robinson(see [9] and [15]), J.E. Roseblade(see [13]), Y.P.Sysak(see [37],[38],[39]and[40]), J.S.Wilson (see [41]), and D.I.Zaitsev(see [11] and [18]).

Now, in this paper, we study the residual finite group and min-by-max subgroups of the group G and its relations, and the end we prove that if finite group G=AB be the product of two subgroups A and B. If A,B, and G are D_π - group, for a set π of primes, then there exist Hall π -subgroups A_0 of A and B_0 of B such that A_0B_0 is a Hall π -subgroups of G.

2. Priliminaries : (elementary properties and theorems.)

In this chapter we express the elementary Lemma and Diffinitons whose used the prove the mantheorem in chapter 3.

2.1. Lemma:

*(*See 25) Let the groups $G=\bigcup_{i\in I}^{t} H_i x_i$, where $H_1,\ldots,H_t$ are subgroups of G. Then at least one of the subgroup H_i has finite index in G.

Proof : Let s be the number of distinct subgroups among $H_1,\ldots,H_t$. If s=1, the lemma is clear. Suppose that s>1, and let I be the set of indices i such that $H_i=H_1$. If $G=\bigcup_{i\in I} H_1 x_i$, then H_i has finite index in G. Assume now that there is an element y in $G\setminus\bigcup_{i\in I} H_1 x_i$. Thus the intersection $H_1 y\cap(\bigcup_{i\in I} H_1 x_i)$ is empty, and hence $H_1 y=\bigcup_{j\notin I} H_j x_j$. Therefore for each i in I we obtain $H_1 x_i\subseteq\bigcup_{j\notin I} H_j x_j y^{-1} x_i$. This proves that G is the union of finitely many cosets of the subgroups H_j, where j is not in I. As the number of distinct subgroups among these is s-1, by induction on s at least one of them has finite index in G.

2.2. Lemma:

Let the group G=AB be the product of two subgroups A and B. If A_0 and B_0 are subgroups of finite index of A and B, respectively, then the subgroup $H=\langle A_0, B_0\rangle$ has index at most mn in G, where $|A:A_0|=m$ and $|B:B_0|=n$.

Proof : Let $\{a_1,\ldots,a_m\}$ be a left transversal of A_0 in A and $\{b_1,\ldots,b_n\}$ a right transversal of B_0 in B. Then.

G=AB= $\bigcup_{i,j} a_i A_0 B_0 b_i=\bigcup_{i,j}(a_i H a_i^{-1})a_i b_i$

is the union of finitely many right cosets of conjugates of H. It follows from Lemma 2.1 that H has finite H has finite index in G. To obtain the required bound for |G:H|, it is clearly enough to consider the finite factor group G/H_G, where H_G is the core of H in G. Consequently we may suppose that G is finite. Then.

$$|G|=\frac{|A|.|B|}{|A\cap B|}\le\frac{|A|.|B|}{|A_0\cap B_0|}=\frac{|A_0|.|B_0|}{|A_0\cap B_0|}mn\le|H|mn,$$ *And so* $|G:H|\le mn.$

2.3. Lemma:

*(*See [1]) Let the group G=AB be the product of two subgroups A and B.

(i) If A and B satisfy the maximal condition on subgroups, then G satisfies the maximal condition on normal subgroups.

(ii) If A and B satisfy the minimal condition on subgroups, then G satisfies the minimal condition on normal subgroups.

Proof: (i) Let $(H_n)_{n\in N}$ be an ascending sequence of normal subgroups of G. Then $(A\cap H_n)_{n\in N}$ and $(B\cap H_n)_{n\in N}$ are ascending sequences of subgroups of A and B, respectively. Hence

$$A\cap H_n=A\cap H_{n+1}\quad and\quad B\cap AH_n=B\cap AH_{n+1}$$

For almost all n. It follows that

$$AH_n=AB\cap AH_n=A(B\cap AH_n)=A(B\cap AH_{n+1})=AH_{n+1}$$

And so

$$H_n=H_n(A\cap H_{n+1})=AH_n\cap H_{n+1}=AH_{n+1}\cap H_{n+1}=H_{n+1}$$

For almost all n. Therefore G satisfies the maximal condition on normal subgroups.

The proof of (ii) is similar.

2.4.Lemma:

Let the group G=AB be the product of two subgroups A and B. If x, y are elements of G, then $G=A^xB^y$. Moreover, there exists an element z of G such that $A^x=A^z$ and $B^y=B^z$.

***Proof* :** Write $xy^{-1}=ab$ with a in A and b in B. If $z=a^{-1}x$, then x=az and $y=b^{-1}z,$ so that $A^x=A^z$ and $B^y=B^z$. It follows that $G= A^z B^z= A^x B^y$.

2.5. Difinition :

Recall that a finite group is a D_π *- groups* if evety $\pi - subgroup$ is contained in a Hall $\pi - subgroup$ and any two Hall $\pi - subgroups$ are conjugate.

2.6. Lemma:

Let the finite group G=AB be the product of two subgroups A and B. If A,B, and G are D_π *- group,* for a set π of primes, then there exist Hall π -subgroups A_0 of A and B_0 of B such that A_0B_0 is a Hall π -subgroups of G.

Proof: Let A_1, B_1, and G_1 be Hall π -subgroups of A, B, and G, respectively. Since G is a D_π *- group,* there exist elements x and y such that A_1^x *and* B_1^y are both contained in G_1. It follows from Lemma 2.4 that $A^x = A^z$ and $B^y = B^z$ for some z in G. Thus $A_0 = A_1^{xz^{-1}}$ and $B_0 = B_1^{yz^{-1}}$ are Hall π -subgroups of A and B, respectively, which are both contained in $G_0 = G_1^{z^{-1}}$. Clearly the order of $A_0 \cap B_0$ is bounded by the *maximum* π -divisor n of the order of $A \cap B$ since $|G| = \frac{|A|.|B|}{|A \cap B|}$, It follows that $|G_0| = \frac{|A_0|.|B_0|}{n} \le \frac{|A_0|.|B_0|}{|A_0 \cap B_0|} = |A_0B_0|$. Therefore $A_0B_0=G_0$ is a Hall π -subgroup of G.

2.7.Corollary:

Let the finite group G=AB be the product of two subgroups A and B .Then for each prime p there exist Sylow p-subgroups A_0 of A and B_0 of B such that A_0B_0 is a Sylow p-subgroup of G.

Proof: *See [5]*

2. 8. Corollary :

Let the finite group G=AB=AK=BK be the product of three nilpotent subgroups, A,B, and K, where K is normal in G. Then G is nilpotent .

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