Special case of Asymptotic Eigenvalues of Second order Differential Equations with Three Turning Points and Neumann conditions

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ABSTRACT: In this present paper I concerned the equation $-W'' + q(x)W = \lambda \phi^2(x)W$, with three turning points and Neumann conditions $W'(0) = W'(1) = 0$. I have obtained the asymptotic eigenvalues when $x \in (x_1, x_2)$.

Keywords: Asymptotic eigenvalues, Turning points, Neumann conditions.

INTRODUCTION

Let consider the boundary value problem of the form

$$-W'' + q(x)W = \lambda \phi^2(x)W \quad \text{for} \quad x \in I = [0, 1] \quad (1)$$

where $\lambda = \rho^2$ is the spectral parameter; $\phi^2(x)$ and $q(x)$ are real functions. We suppose that

$$\phi^2(x) = \prod_{i=1}^{3} (x-x_i)^{l_i} \phi_0(x) \quad (2)$$

where, $0 < x_1 < x_2 < x_3 < 1$, $l_i \in \mathbb{N}$, $\phi_0(x) > 0$ for $x \in I = [0, 1]$ and $\phi_0(x)$ is twice continuously differentiable function on $I$. On other words, $\phi^2(x)$ has in three zeros $x_i$, $i = 1, 2, 3$ of order $l_i$, the zeros $x_i$ of $\phi^2(x)$ are called turning points. In this section we obtained the asymptotic eigenvalues of equation (1) in three turning points with Neumann conditions $W'(0) = W'(1) = 0$.

Notations

In the real second-order differential equation

$$-W'' + q(x)W = \lambda \phi^2(x)W \quad \text{for} \quad x \in I = [0, 1] \quad (3)$$

$\phi^2(x)$ has in $I$, there zeros $x_i$ of order $l_i$, $i = 1, 2, 3$ where, $l_1$ is even, $l_2$ is odd and $l_3$ is even. Let $\epsilon > 0$ be fixed sufficiently small and let
The fundamental system solutions for \( x \in I = [0, 1] \)

Now, let \( W(x, \lambda) \) be the solution of equation (1). The fundamental system of solutions (FSS) for equation (1), when \( x_1 \) can be represented in the form (see [1] page 219) since \( x_1 \) is type of \( I \), we have the following FSS for \( \rho \in S_{-1} \)

The fundamental system solutions for \( x \in I = [0, 1] \)

We use for convenience the abbreviations

The sectors \( S_{-1} \) is in form of

We know from [1] that \( x_1 \) is type of \( I \), \( x_2 \) is type of \( IV \) and \( x_3 \) is of type \( II \).

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The FSS of (1) for \( x_1 \) that is type of I whit sector \( S_{-1} \) are the following form
The boundary value problem

$$W_{1,1}(x, \rho) = \begin{cases} \phi(x) \left( \frac{1}{2} e^{\rho \int_{x_1}^{x} \phi(t) dt} \right) & 0 \leq x < x_1 \\ \phi(x) \left( \frac{1}{2} \csc \pi \mu e^{\rho \int_{x}^{x_1} \phi(t) dt} \right) & x_1 < x < x_2 \end{cases}$$

$$W_{2,1}(x, \rho) = \begin{cases} \phi(x) \left( \frac{1}{2} e^{-\rho \int_{x}^{x_1} \phi(t) dt} \right) & 0 \leq x < x_1 \\ \phi(x) \left( \frac{1}{2} \sin \pi \mu e^{-\rho \int_{x_1}^{x} \phi(t) dt} \right) & x_1 < x < x_2, \end{cases}$$

The Wronskian of FSS satisfies in following form

$$W(\rho) = W(W_{1,1}(x, \rho), W_{2,1}(x, \rho)) = -2\rho[1].$$

**Asymptotic form of the solutions for**

$$W'(0) = W'(1) = 0$$

Let us consider the differential equation (1) with following boundary conditions

$$C(0, \lambda) = 1, C'(0, \lambda) = 0.$$ (9)

By applying the FSS $$V_{1,1}(x, \rho), V_{2,1}(x, \rho)$$ for $$x \in I_{1,\epsilon}$$ we have

$$C(x, \rho) = c_1 W_{1,1}(x, \rho) + c_2 W_{2,1}(x, \rho).$$

By derivation from $$C(x, \rho)$$ we can write

$$C'(x, \rho) = c_1 W_{1,1}'(x, \rho) + c_2 W_{2,1}'(x, \rho) \quad \text{for} \quad x \in I_{1,\epsilon}.$$ (10)

We infer by using Cramer's rule leads to the following equation

$$C(x, \rho) = \frac{1}{W(\rho)} \left(W_{1,2}(x, \rho)W_{2,1}'(0, \rho) - W_{1,2}'(0, \rho)W_{2,1}(x, \rho)\right)$$ (11)

where, $$W(\rho) = -2\rho[1].$$

**Derivative of solutions and asymptotic eigenvalues**

Let us consider boundary value problem $$L_1 = L_1(\phi, q(x), b)$$ for equation (1) with boundary conditions

$$C(0, \lambda) = 1, C'(b, \lambda) = 0, C'(0, \lambda) = 0.$$ (12)

The boundary value problem $$L_1$$ for $$b \in (x_1, x_2)$$ has a countable set of positive eigenvalues.

Now for fixed $$x \in (x_1, x_2)$$ and use (7),(8) we determine the connection coefficients $$T_1(\rho), T_2(\rho)$$
\[ C(x, \rho) = T_1(\rho)W_{1,1} + T_2(\rho)W_{2,1} \Rightarrow C'(x, \rho) = T_1(\rho)W'_{1,1} + T_2(\rho)W'_{2,1}. \]  

(13)

For, \( x_1 < x < x_2 \Rightarrow \)

\[
\begin{align*}
W_{1,1}(x, \rho) &= \left[ \phi(x) \right]^{\frac{1}{2}} \csc \pi \mu e^{-\rho \int_{\phi(0)}^{\phi(x)} |\phi'| dt} \\
W_{2,1}(x, \rho) &= \left[ \phi(x) \right]^{\frac{1}{2}} \sin \pi \mu e^{-\rho \int_{\phi(0)}^{\phi(x)} |\phi'| dt}
\end{align*}
\]

(14)

Let consider,

\[ A = e^{\rho \int_{\phi(0)}^{\phi(x)} |\phi'| dt} \]  

[1] \( \Rightarrow \)

\[ W_{1,2}(x, \rho) = \left[ \phi(x) \right]^{\frac{1}{2}} \csc \pi \mu A \]

The derivative of \( W_{1,2}(x, \rho) \) is following form

\[ W'_{1,2}(x, \rho) = \csc \pi \mu \mu \left( \left[ \phi(x) \right]^{\frac{1}{2}} A' + \left[ \phi(x) \right]^{\frac{1}{2}} A' \right) \]

By use of fundamental theorem the derivative of A is as follow

\[
\begin{align*}
A' &= \rho \phi(x) \left[ e^{\rho \int_{\phi(0)}^{\phi(x)} |\phi'| dt} \right] = \rho \phi(x) B \\
B &= e^{\rho \int_{\phi(0)}^{\phi(x)} |\phi'| dt - \frac{\pi}{2}}[1]
\end{align*}
\]

(15)

So \( W'_{1,2}(x, \rho) \) is in following form

\[
\begin{align*}
W_{1,2}'(x, \rho) &= \csc \pi \mu \left( \left[ \phi(x) \right]^{\frac{1}{2}} A' + \rho \phi(x) B \right) \\
C &= \left[ \phi(x) \right]^{\frac{1}{2}} A, D = \phi(x) \Rightarrow W_{1,2}'(x, \rho) = \rho \csc \pi \mu \left( \frac{1}{\rho} C + A \right) \Rightarrow \end{align*}
\]

(16)

\[
\begin{align*}
W_{1,2}'(x, \rho) &= \rho \csc \pi \mu DB(1 + \frac{1}{\rho} CA) \Rightarrow W_{1,2}'(x, \rho) = \rho \csc \pi \mu DB \left( 1 + O\left( \frac{1}{\rho} \right) \right)
\end{align*}
\]

At last

\[
\begin{align*}
W_{1,2}'(x, \rho) &= \rho \csc \pi \mu DB \left( 1 + O\left( \frac{1}{\rho} \right) \right) \\
W_{1,2}'(x, \rho) &= \rho \csc \pi \mu \left[ \phi(x)e^{\rho \int_{\phi(0)}^{\phi(x)} |\phi'| dt} \right] [1] \left( 1 + O\left( \frac{1}{\rho} \right) \right)
\end{align*}
\]

(17)

Similarly the derivative of \( W_{2,1}(x, \rho) \) is as following form so, Let us suppose
\[ M = e^{-\rho \int_{x}^{y} \| \phi(t) \| dt} \quad [1] \Rightarrow M' = -\rho \| \phi(x) \| e^{-\rho \int_{x}^{y} \| \phi(t) \| dt} \quad [1] = -\rho \| \phi(x) \| M \Rightarrow W_{2,1}(x, \rho) = \| \phi(x) \| \frac{1}{\rho} \sin \pi \mu_{1} M \]

\[
\begin{align*}
W'_{2,1}(x, \rho) &= \sin \pi \mu_{1} \left( \| \phi(x) \| \frac{1}{\rho^2} M + \| \phi(x) \| \frac{1}{2} M' \right) = \\
\sin \pi \mu_{1} \left( KM - M \| \phi(x) \| \frac{1}{\rho} \right) &= \sin \pi \mu_{1} \left( -\frac{1}{\rho} KM - ML \right) \\
K &= (\| \phi(x) \| \frac{1}{\rho^2})', L = -\| \phi(x) \| \frac{1}{\rho} \\
W'_{2,1}(x, \rho) &= \rho \sin \pi \mu_{1} (ML) \left( 1 - \frac{1}{\rho} \frac{K}{L} \right) = \rho \sin \pi \mu_{1} (ML) \left( 1 + O(\frac{1}{\rho}) \right)
\end{align*}
\]

Finally, \( W'_{2,1}(x, \rho) \) is in form

\[
W'_{2,1}(x, \rho) = -\rho \sin \pi \mu_{1} (ML) \left( 1 - \frac{1}{\rho} \frac{K}{L} \right) = -\rho \sin \pi \mu_{1} (e^{-\rho \int_{x}^{y} \| \phi(t) \| dt}) \left( 1 + O(\frac{1}{\rho}) \right)
\]

Hence we have estimated the solution of (1) defined by the initial condition (9) and Cramer's rule to determine the connection coefficients \( T_{1}(\rho), T_{2}(\rho) \) with

\[
\begin{align*}
C(x, \rho) &= T_{1}(\rho)W_{1,2}(x, \rho) + T_{2}(\rho)W_{2,1}(x, \rho) \Rightarrow 1 = T_{1}(\rho)W_{1,2}(0, \rho) + T_{2}(\rho)W_{2,1}(0, \rho) \\
C'(x, \rho) &= T_{1}(\rho)W'_{1,2}(x, \rho) + T_{2}(\rho)WW'_{2,1}(x, \rho) \Rightarrow 0 = T_{1}(\rho)W'_{1,2}(0, \rho) + T_{2}(\rho)W'_{2,1}(0, \rho)
\end{align*}
\]

\[
\begin{align*}
T_{1}(\rho) &= \frac{W'_{2,1}(0, \rho)}{-2\rho[1]} = -\sin \pi \mu_{1} \| \phi(0) \| \frac{1}{\rho} (e^{-\rho \int_{x}^{y} \| \phi(t) \| dt}) \left( 1 + O(\frac{1}{\rho}) \right) \\
T_{2}(\rho) &= \frac{W'_{1,2}(0, \rho)}{2\rho[1]} = \frac{1}{4} \| \phi(0) \| \frac{1}{\rho} \csc \pi \mu_{1} (e^{\rho \int_{x}^{y} \| \phi(t) \| dt}) \left( 1 + O(\frac{1}{\rho}) \right)
\end{align*}
\]

\[
T_{2}(\rho) = \frac{1}{4} \| \phi(0) \| \frac{1}{\rho} \csc \pi \mu_{1} \Gamma_{1}, T_{1}(\rho) = -\sin \pi \mu_{1} \| \phi(0) \| \frac{1}{\rho} \Gamma_{2}
\]

\[
\begin{align*}
\Gamma_{1} &= (e^{\rho \int_{x}^{y} \| \phi(t) \| dt})^{[1]} \Gamma_{4} = (e^{\rho \int_{x}^{y} \| \phi(t) \| dt})^{[1]} \left( 1 + O(\frac{1}{\rho}) \right) \\
\Gamma_{2} &= (e^{-\rho \int_{x}^{y} \| \phi(t) \| dt})^{[1]} \\
\Gamma_{3} &= (e^{\rho \int_{x}^{y} \| \phi(t) \| dt})^{[1]} - \left( e^{-\rho \int_{x}^{y} \| \phi(t) \| dt} \right)^{[1]} \left( 1 + O(\frac{1}{\rho}) \right), \Gamma_{2} = (e^{-\rho \int_{x}^{y} \| \phi(t) \| dt})^{[1]}
\end{align*}
\]

\[
W'_{2,1}(x, \rho) = -\rho \sin \pi \mu_{1} \| \phi(x) \| \frac{1}{\rho} \Gamma_{4}, W'_{1,2}(x, \rho) = \frac{1}{2} \rho \csc \pi \mu_{1} \| \phi(x) \| \frac{1}{\rho} \Gamma_{3}
\]

By substituting (23) and (22) in (20) we obtain the leading term as \( C(x, \rho) \) follows
\[
C'(x, \rho) = -\frac{1}{4}[\phi(0)]^2 \csc \pi \mu_1 \Gamma_1 2i \rho \sin \pi \mu_1 |\phi(x)| \frac{1}{2} \Gamma_4 - \frac{1}{2} \sin \pi \mu_1 |\phi(0)| \frac{1}{2} \Gamma_2 i \rho \csc \pi \mu_1 |\phi(x)| \frac{1}{2} \Gamma_3 = \frac{1}{2} \sin \pi \mu_1 |\phi(0)| \frac{1}{2} \Gamma_2 \rho \csc \pi \mu_1 |\phi(x)| \frac{1}{2} \Gamma_3 + \frac{1}{2} |\phi(0)|^2 \csc \pi \mu_1 \Gamma_1 \rho \sin \pi \mu_1 |\phi(x)| \frac{1}{2} \Gamma_4 \tag{24}
\]

Now must to determine the value of \( \Gamma_2 \Gamma_3 \) and \( \Gamma_1 \Gamma_4 \)

\[
\begin{align*}
\Gamma_2 \Gamma_3 &= (e^{-\rho \int_0^1 |\phi(x)| dx}) \left( e^{\rho \int_0^1 |\phi(x)| dx} [1] - e^{-\rho \int_0^1 |\phi(x)| dx + \frac{\pi}{2}} [1] \right) \left( 1 + O\left( \frac{1}{\rho} \right) \right), \\
\Gamma_1 \Gamma_4 &= (e^{-\rho \int_0^1 |\phi(x)| dx}) \left( e^{\rho \int_0^1 |\phi(x)| dx} - e^{-\rho \int_0^1 |\phi(x)| dx + \frac{\pi}{2}} \right) \left( 1 + O\left( \frac{1}{\rho} \right) \right) \tag{25}
\end{align*}
\]

By applying \( C'(x, \rho) = 0 \), consequently \( \Gamma_2 \Gamma_3 + \Gamma_1 \Gamma_4 = 0 \)

\[
\begin{align*}
\Gamma_2 \Gamma_3 &= -\Gamma_1 \Gamma_4 \Rightarrow (e^{-\rho \int_0^1 |\phi(x)| dx}) \left( \frac{e^{\rho \int_0^1 |\phi(x)| dx}}{e^{\rho \int_0^1 |\phi(x)| dx} - e^{-\rho \int_0^1 |\phi(x)| dx + \frac{\pi}{2}}} \right) = \\
&= \left( (1 + O\left( \frac{1}{\rho} \right) \right)
\end{align*}
\]

\[
\begin{align*}
[1] &= \frac{1 - e^{2\rho \int_0^1 |\phi(x)| dx}}{e^{2\rho \int_0^1 |\phi(x)| dx} - 1} = 1 + O\left( \frac{1}{\rho} \right) \Rightarrow K = 1 - e^{2\rho \int_0^1 |\phi(x)| dx} \\
\frac{K}{e^{2\rho \int_0^1 |\phi(x)| dx} - 1} &= 1 + O\left( \frac{1}{\rho} \right) \Rightarrow \frac{K}{1 + O\left( \frac{1}{\rho} \right)} = e^{2\rho \int_0^1 |\phi(x)| dx} - 1 \\
K + O\left( \frac{1}{\rho} \right) &= e^{2\rho \int_0^1 |\phi(x)| dx} - 1 \Rightarrow K + 1 + O\left( \frac{1}{\rho} \right) = e^{2\rho \int_0^1 |\phi(x)| dx} \tag{26}
\end{align*}
\]
\[ \ln \left( K + 1 + O \left( \frac{1}{\rho} \right) \right) = O \left( \frac{1}{\rho^2} \right) = 2 \rho \int_{x_1}^{x_2} |\phi(t)| dt - i \frac{\pi}{4} - 2k \bar{\alpha} \] (27)

By dividing (27) by \[ \int_{x_1}^{x_2} |\phi(t)| dt \] I obtained the leading term \( \rho \) as follows

\[ \rho_k = \frac{i \frac{\pi}{4} + 2k \bar{\alpha}}{2 \int_{x_1}^{x_2} |\phi(t)| dt} + O \left( \frac{1}{\rho^2} \right) = \frac{k \pi + \frac{\pi}{8}}{\int_{x_1}^{x_2} |\phi(t)| dt} + O \left( \frac{1}{\rho^2} \right) \]

\[ REFERENCES \]


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